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The Nonabelian Simple Groups G, $|G| < 10^6$ — Maximal Subgroups*

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Abstract. The maximal subgroups of all the simple groups (except $L(2, q)$) of order up to one million are given to within conjugacy. Permutation characters on the cosets of the maximal subgroups are given, as are orbit lengths (whenever practical).

Introduction. We present a complete list, to within conjugacy, of the maximal subgroups of each nonabelian simple group G, $|G| > 10^6$, excepting the family $L(2, q)$ for which the subgroups are described in the literature [10].

For each G we calculate its permutation character on the cosets of each maximal subgroup and the corresponding orbits of the representation restricted to that subgroup. Use was made of the GROUP program [2] on a CDC 6400 computer.

Notation. Repeated use is made of the character tables for the simple groups G, $|G| < 10^6$, found in [13]. We refer to the irreducible characters of a group by their degrees and, if necessary, a subscript determined by their order of appearance in these character tables; for example, the irreducible characters of M_{22} are denoted by 1, 21, 45_1, 45_2, 55, 99, 154, 210, 231, 280_1, 280_2, 385.

We use the following notation:

$N(K) = N_G(K)$ is the normalizer of K in G;

$C(K) = C_G(K)$ is the centralizer of K in G;

$|H|$ is the order of H;

χ_i denotes the ith irreducible character of G;

ϕ_H is a permutation character of G on the cosets of a subgroup H;

2A, 3B, etc., refer to conjugacy classes of G as listed in the character tables in [13];

n is the cyclic group of order n;

Σ_n is the symmetric group of degree n;

A_n is the alternating group of degree n;

S_p is a Sylow p-subgroup;

$A.B$ is an extension of A by B;

$A.B.C = A.(B.C)$;

p^n is the elementary abelian group of order p^n;

$(m, n)^+$ is the even subgroup of index two in $\Sigma_m \times \Sigma_n$.

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The Subgroups and Permutation Characters. Since more than 100 conjugacy classes of maximal subgroups occur, it is impractical to indicate proofs, but we do give the useful propositions.

Some Useful Results. The following are repeatedly used in identifying maximal subgroups:

PROPOSITION 1 (FINKELSTEIN [6]). *Any maximal subgroup of a simple group G is the normalizer in G of a direct product of isomorphic simple groups. In particular, a maximal subgroup not containing a nonabelian simple group must normalize an elementary abelian group.*

PROPOSITION 2. *If* $p^{\alpha} \mid |G|$ *and* $H = N_G(K)$ *where* $|K| = p^{\beta}$, $0 < \beta < \alpha$, *then* $p^{\beta+1} \mid |H|$.

Proof. Clearly,

$$H = N_G(K) \geqslant N_{S_p}(K).$$

Since $K \nleqslant S_p$ and S_p is nilpotent, we have $K \neq N_{S_p}(K)$. Thus

$$K \nleqslant N_{S_p}(K) \leqslant N_G(K),$$

so $p^{\beta+1} \mid |N_G(K)|$.

The following were used to find the irreducible constituents of ϕ_H.

PROPOSITION 3. *A permutation group is primitive if and only if it is equivalent to a representation on the cosets of a maximal subgroup.*

PROPOSITION 4.

$$\phi_H(x) = \frac{|H \cap x^G|.|C_G(x)|}{|H|}.$$

In the following f_j is the degree of an irreducible constituent of ϕ_H, e_j its multiplicity, n_i the length of an orbit of H, k the number of orbits, and $n = [G : H]$.

PROPOSITION 5 (FRAME [19]). *The expression of*

$$q = n^{k-2}\prod_i n_i \Big/ \prod_j f_j e_j^2$$

is a rational integer. If, in addition, all irreducible constituents of ϕ_H *are rational, then q is a square.*

PROPOSITION 6 (SEE [19]). *If all* $e_j = 1$ *and l of the numbers* f_j *are divisible by some* p^{α}, *then at least l of the numbers* nn_i *are divisible by* p^{α}.

The Alternating Groups. We do an analysis of the maximal subgroups of the alternating groups.

(i) *Intransitive subgroups.* By $(k, l)^+$ where $k + l = n$ we mean $A_n \cap (\Sigma_k \times \Sigma_l)$, whose transitive constituents have lengths k and l. (Notice that a subgroup of A_n having > 2 transitive constituents cannot be maximal.) Now $(k, l)^+$ is maximal whenever $k \neq l$; this may be proved by induction on n. It is easy to see that $(k, l)^+ \cong (A_k \times A_l).2$.

(ii) *Imprimitive subgroups.* The group $H = (k, k, \ldots, k)^+$, $k > 1$, is not maximal since we may always adjoin an element to H giving rise to a group M for which $H \nleq M \nleq A_n$ and M is maximal imprimitive. The order of M is $\frac{1}{2}b!(k!)^n$, where b is the number of transitive constituents of H. This may be shown by an elementary counting argument. The subgroup M is maximal except when $k = 2$—again this is shown by induction. When $k = 2$, A_8 contains an example of M not maximal: $(2, 2, 2, 2)^+$ is a group of order 192 which is contained in $2^3.L(3, 2)$.

(iii) *Primitive subgroups.* Sims' list [17] gives the primitive groups of degree up to 20 to within isomorphism. Those of degree n which are maximal subgroups of A_n are determined by examining orders or by arguments using Propositions 1 and 2 and/or the character table for A_n.

The number of conjugacy classes of each type of maximal subgroup is determined in the following manner:

(i) *Intransitive.* $(k, l)^+$, $k \neq l$. Each of these groups is the stabilizer of a point in the (transitive) action of A_n on the k-tuples of $\{1, \ldots, n\}$. Thus, there is one class each of the intransitive maximal subgroups.

(ii) *Imprimitive.* The action of A_n on the k-tuples of $\{1, \ldots, n\}$ is imprimitive, and the maximal imprimitive group containing $(k, k, \ldots, k)^+$ is the stabilizer of a point in the action of A_n on a resulting block system. Again, we have only one class of each maximal subgroup.

(iii) *Primitive.* The number of classes of the primitive maximal subgroups was arrived at using structure constants for the classes of involutions in [13] and, in some cases, characterizations of the simple groups as minimal $(2, m, n)$ groups in [20].

We note that the four-dimensional linear groups including $L(4, 2) \cong A_8$ have recently been treated by Mwene [16], who classifies the maximal subgroups up to isomorphism.

Finally, the authors would like to learn of any inaccuracies found in the tables.

Order	Index	Description	Permutation Character	Orbit Lengths
		A_5 $(60 = 2^2.3.5)$		
6	10	$\Sigma_3 = (3, 2)^+$, intransitive	1+4+5	1, 3, 6
10	6	dihedral, primitive	1+5	1, 5
12	5	$A_4 = (4, 1)^+$, intransitive	1+4	1, 4
		$L(3, 2)$ $(168 = 2^3.3.7)$		
21	8	7.3	1+7	1, 7
24	7	Σ_4	1+6	1, 6
24	7	Σ_4	1+6	1, 6

Order	Index	Description	Permutation Character	Orbit Lengths
		A_6 $(360 = 2^3.3^2.5)$		
24	15	$\Sigma_4 = (4, 2)^+$, intransitive	$1+5_1+9$	1, 6, 8
24	15	Σ_4, imprimitive	$1+5_2+9$	1, 6, 8
36	10	$(A_3 \times A_3).2.2$	$1+9$	1, 9
60	6	$A_5 = (5, 1)^+$, intransitive	$1+5_1$	1, 5
60	6	A_5, transitive	$1+5_2$	1, 5
		A_7 $(2520 = 2^3.3^2.5.7)$		
72	35	$(4, 3)^+$, intransitive	$1+6+14_1+14_2$	1, 4, 12, 18
120	21	$\Sigma_5 = (5, 2)^+$, intransitive	$1+6+14_2$	1, 10, 10
168	15	$L(3, 2)$, primitive	$1+14_1$	1, 14
360	7	$A_6 = (6, 1)^+$, intransitive	$1+6$	1, 6
		$L(3, 3)$ $(5616 = 2^4.3^3.13)$		
24	234	$\Sigma_4 = N(2^2)$	$1+2.12+16_1+16_2$ $+16_3+16_4+2\times 26_1$ $+2.27+39$	$1, 3, 4^2, 6, 12^8, 24^5$
39	144	$13.3 = N(13)$	$1+13+16_1+16_2+16_3$ $+16_4+27+39$	$1, 13^5, 39^2$
432	13	Hessian.2	$1+12$	1, 12
432	13	Hessian.2	$1+12$	1, 12
		$U(3, 3)$ $(6048 = 2^5.3^3.7)$		
96	63	$C(2A)$	$1+14+21_1+27$	1, 6, 24, 32
96	63	$N(2^2)$	$1+7_2+7_3+21+27$	$1, 6, 16^2, 24$
168	36	$L(3, 2)$	$1+7_2+7_3+21_1$	1, 7, 7, 21
216	28	Hessian	$1+27$	1, 27
		M_{11} $(7920 = 2^4.3^2.5.11)$		
48	165	$M_8.\Sigma_3 = C(2A)$	$1+10_1+11+2.44+55$	$1, 8, 12, 24^4, 48$
120	66	$\Sigma_5 = N(A_5)$	$1+10_1+11+44$	1, 15, 20, 30
144	55	$M_9.2 = N(3^2)$	$1+10_1+44$	1, 18, 36
660	12	$L(2, 11)$	$1+11$	1, 11
720	11	M_{10}	$1+10_1$	1, 10

Order	Index	Description	Permutation Character	Orbit Lengths
		A_8 (20, 160 = $2^6.3^2.5.7$)		
360	56	$(5, 3)^+$, intransitive	1+7+20+28	1, 10, 15, 30
576	35	$(A_4 \times A_4).2.2$, imprimitive	1+14+20	1, 16, 18
720	28	$\Sigma_6 = (6, 2)^+$, intransitive	1+7+20	1, 12, 15
1344	15	$2^3.L(3, 2)$, primitive	1+14	1, 14
1344	15	$2^3.L(3, 2)$, primitive	1+14	1, 14
2520	8	$A_7 = (7, 1)^+$, intransitive	1+7	1, 7
		$L(3, 4)$ (20, 160 = $2^6.3^2.5.7$)		
72	280	$N(S_3)$	$1+20+45_1+45_2+35_1+35_2+35_3+64$	$1, 9, 18^3, 72^3$
168	120	$L(3, 2)$	$1+20+35_1+64$	1, 21, 42, 56
168	120	$L(3, 2)$	$1+20+35_2+64$	1, 21, 42, 56
168	120	$L(3, 2)$	$1+20+35_3+64$	1, 21, 42, 56
360	56	A_6	$1+20+35_1$	1, 10, 45
360	56	A_6	$1+20+35_2$	1, 10, 45
360	56	A_6	$1+20+35_3$	1, 10, 45
960	21	4^2.PGL(2, 4)	1+20	1, 20
960	21	4^2.PGL(2, 4)	1+20	1, 20
		$U(4, 2)$ (25, 920 = $2^6.3^4.5$)		
576	45	$C(2A)$	1+20+24	1, 12, 32
648	40	$C(3A)$	$1+15_1+24$	1, 12, 27
648	40	$N(3^2)$	$1+15_1+24$	1, 12, 27
720	36	Σ_6	$1+15_2+20$	1, 15, 20
960	27		1+6+20	1, 10, 16
		Sz(8) (29, 120 = $2^6.5.7.13$)		
20	1456	$N(5)$	$1+2.35_1+2.35_2+2.35_3+4.65_1+4.65_2+4.65_3+3.64+3.91$	
52	560	$N(13)$	$1+35_1+35_2+35_2+64+2.65_1+2.65_2+2.65_3$	$1, 13, 26^9, 52^6$
448	65		1+64	1, 64

Order	Index	Description	Permutation Character	Orbit Lengths
		$U(3, 4)$ $(62{,}400 = 2^6.3.5^2.13)$		
39	1600	$N(13)$	$1+39_1+39_2+2.52_1$ $+2.52_2+2.52_3+2.52_4$ $+2.64+65_1+65_2+65_3$ $+65_4+65_5+13_1+13_2$ $+13_3+13_4+2.75_1$ $+2.75_2+2.75_3+2.75_4$	
150	416	$(5^2).\Sigma_3$	$1+39_1+39_2+52_1+52_2$ $+52_3+52_4+64+65_1$	
300	208	$5 \times A_5$	$1+39_1+39_2+64+65_1$	
960	65		$1+64$	1, 64
		M_{12} $(95{,}040 = 2^6.3^3.5.11)$		
72	1320	$A_4 \times \Sigma_3 = N(2^2)$	$1+16_1+16_2+2.45 + 2.54$ $+55_3+2.66+2.99$ $+2.120+2.144+176$	
192	495	$2^2.2^3.\Sigma_3 = N(2^2)$	$1+16_1+16_2+45+2.54$ $+66+99+144$	$1, 6, 16, 24, 32^2, 48^2, 96^3$
192	495	$M_8.\Sigma_4 = C(2B)$	$1+11_1+11_2+55_3$ $+2.54+66+99+144$	$1, 6, 16, 24, 32^2, 48^2, 96^3$
240	396	$2 \times \Sigma_5 = C(2A)$	$1+16_1+16_2+45$ $+2.54+66+144$	
432	220	$M_9.\Sigma_3$ = Hessian .2	$1+11_1+54+55_3+99$	1, 9, 18, 48, 144
432	220	$M_9.\Sigma_3$ = Hessian .2	$1+11_2+54+55_3+99$	1, 9, 18, 48, 144
660	144	$L(2, 11)$	$1+11_1+11_2+55_3+66$	$1, 11^2, 55, 66$
1440	66	$M_{10}.2$	$1+11_1+54$	1, 45, 20
1440	66	$M_{10}.2$	$1+11_2+54$	1, 45, 20
7920	12	M_{11}	$1+11_1$	1, 11
7920	12	M_{11}	$1+11_2$	1, 11
		$U(3, 5)$ $(126{,}000 = 2^4.3^2.5^3.7)$		
240	525	$C_6(2A)$	$1+28_1+28_2+28_3+84$ $+105+125+126_1$	
720	175	M_{10}	$1+21+28_1+125$	1, 12, 72, 90
1000	126	$N(5A)$	$1+125$	1, 125
2520	50	A_7	$1+21+28_1$	1, 7, 42
2520	50	A_7	$1+21+28_2$	1, 7, 42
2520	50	A_7	$1+21+28_3$	1, 7, 42

Order	Index	Description	Permutation Character	Orbit Lengths
		J_1 (175,560 = $2^3.3.5.7.11.19$)		
42	4180	$N(7)$, Frobenius group	$1+3.120_1+3.120_2+3.120_3+2.56_1+2.56_2+5.209+6.133_1+4.77_1+4.76_1+77_2+77_3+133_2+133_3$	
60	2926	15.2^2	$1+4.209+2.120_1+2.120_2+2.120_3+77_1+2.76_1+76_2+2.133_1+56_1+56_2+2.133_2+2.133_3+77_2+77_3$	
110	1596	$N(11)$, Frobenius group	$1+120_1+120_2+120+2.209+2.77_1+76_2+76_3+133_1+133_2+133_3+56_1+56_2$	
114	1540	$N(19)$, Frobenius group	$1+2.209+2.133_1+77_1+77_2+77_3+2.76_1+56_1+56_2+120_1+120_2+120_3$	
120	1463	$2 \times A_5 = C(2A)$	$1+56_1+56_2+2.76_1+2.77_1+120_1+120_2+120_3+2.133_1+2.209$	
168	1045	$2^3.7.3$	$1+120_1+120_2+120_3+209+133_1+76_1+56_1+56_2+77_2+77_3$	
660	266	$L(2, 11)$	$1+56_1+56_2+76_1+77_1$	1, 11, 12, 110, 132
		A_9 (181,440 = $2^6.3^4.5.7$)		
216	840	Hessian, primitive	$1+21_1+21_2+48+56+2.84+120+189+216$	
648	280	imprimitive	1+27+48+84+120	1, 27, 36, 54, 162
1440	126	$(5, 4)^+$, intransitive	1+8+27+42+48	1, 5, 20, 40, 60
1512	120	$L(2, 8).3$, primitive	$1+35_1+84$	1, 56, 63
1512	120	$L(2, 8).3$, primitive	$1+35_2+84$	1, 56, 63
2160	84	$(6, 3)^+$, intransitive	1+8+27+48	1, 18, 20, 45
5040	36	$\Sigma_7 = (7, 2)^+$, intransitive	1+8+27	1, 14, 21
20160	9	$A_8 = (8, 1)^+$, intransitive	1+8	1, 8

Order	Index	Description	Permutation Character	Orbit Lengths
		$L(3, 5)$ (372,000 = $2^5.3.5^3.31$)		
93	4000	$N(31)$	$1+31_1+31_2+31_3$ $+\Sigma_1^{10}96_i+125$ $+155_1+155_2$ $+155_3+2.186$ $+2.124_1+2.124_2$ $+2\Sigma_5^{10}124_i$	$1, 31^{15}, 93^{38}$
96	3875	Monomial group	$1+\Sigma_1^{10}96_i+2.30+31_1$ $+2.124_1+3.124_2$ $+2.124_5+2.124_6$ $+124_7+124_8$ $+124_9+124_{10}$ $+2.125+3.155_1$ $+155_2+155_3+186$	
120	3100	$N(2^2) = SL(2, 5)$	$1+2.30+2.31_1+2.31_2$ $+2.31_3+\Sigma_1^{10}124_i+125$ $+2.155_1+2.155_2$ $+2.155_3+3.186$	
12000	31	$5^2.GL(2, 5)$	1+30	1, 30
12000	31	$5^2.GL(2, 5)$	1+30	1, 30
		M_{22} (443,520 = $2^7.3^2.5.7.11$)		
660	672	$L(2, 11)$	1+21+55+154+210 +231	$1, 55^2, 66$ 165, 330
720	616	M_{10}	1+21+55+154+385	1, 30, 45, 180, 360
1344	330	$2^3.L(3, 2) = N(2^3)$	1+21+55+99+154	$1, 21, 84, 112^2$
1920	231	$2^4.\Sigma_5 = N(2^4)$	1+21+55+154	1, 30, 40, 160
2520	176	A_7	1+21+154	1, 70, 105
2520	176	A_7	1+21+154	1, 70, 105
5760	77	$2^4.A_6$	1+21+55	1, 16, 60
20160	22	M_{21}	1+21	1, 21
		J_2 (604,800 = $2^7.3^3.5^2.7$)		
60	10080	A_5	$1+14_1+14_2+36+63$ $+70_1+70_2+3.90$ $+4.126+4.160+2.175$ $+3.189_1+3.189_2$ $+2.224_1+2.224_2$ $+6.225+4.288+5.300$ $+6.336$	

Order	Index	Description	Permutation Character	Orbit Lengths
300	2016	$5^2.(2 \times \Sigma_3)$	1+63+2.90+2.126 +160+175+225+288 +2.336	
336	1800	$N(L(2, 7))$	1+36+2.63+90+2.126 +160+175+288+2.336	
600	1008	$A_5 \times D_{10}$	$1+14_1+14_2+2.90+126$ +160+225+288	
720	840	$A_4 \times A_5$	1+63+90+126+160 +175+225	
1152	525	$(2^6.3)\Sigma_3$	1+36+90+160+63 +175	
1920	315	$2^5.A_5$	$1+36+90+160+14_1$ $+14_2$	$1, 10, 32^2,$ 80, 160
2160	280	3.PGL(2, 9)	1+63+90+126	1, 36, 108, 135
6048	100	$U(3, 3)$	1+36+63	1, 36, 63
		Sp(4, 4) ($979,200 = 2^8.3^2.5^2.17$)		
720	1360	$Sp(4, 2) \cong \Sigma_6$	$1+2.50+85_1+85_2$ $+153+256+340_1+340_2$	
7200	136		$1+50+85_1$	1, 60, 75
7200	136		$1+50+85_2$	1, 60, 75
8160	120	$L(2, 16).2$	$1+34_1+85_1$	1, 51, 68
8160	120	$L(2, 16).2$	$1+34_2+85_2$	1, 51, 68
11520	85		$1+34_1+50$	1, 20, 64
11520	85		$1+34_2+50$	1, 20, 64

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